

# Towards Noncommutative Linking Numbers Via the Seiberg-Witten Map

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## Abstract

In the present work some geometric and topological implications of noncommutative Wilson loops are explored via the Seiberg-Witten map. In the abelian Chern-Simons theory on a three dimensional manifold, it is shown that the effect of noncommutativity is the appearance of  $6^n$  new knots at the  $n$ -th order of the Seiberg-Witten expansion. These knots are trivial homology cycles which are Poincaré dual to the high-order Seiberg-Witten potentials. Moreover the linking number of a standard 1-cycle with the Poincaré dual of the gauge field is shown to be written as an expansion of the linking number of this 1-cycle with the Poincaré dual of the Seiberg-Witten gauge fields. In the process we explicitly compute the noncommutative 'Jones-Witten' invariants up to first order in the noncommutative parameter. Finally in order to exhibit a physical example, we apply these ideas explicitly to the Aharonov-Bohm effect. It is explicitly displayed at first order in the noncommutative parameter, we also show the relation to the noncommutative Landau levels.

*Key words:* Seiberg-Witten map; Wilson loop; topological field theory; knot invariants.

September 1, 2015

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# 1 Introduction

Noncommutativity of spacetime has strongly attracted the attention in the last two decades (see for instance [1, 2]). In a remarkable paper N. Seiberg and E. Witten [3], they showed that open string theory in the specific limit of small volume, non-vanishing  $B$ -field and  $\alpha' \rightarrow 0$ , the amplitudes with constant open string metric  $G$  and noncommutativity parameter  $\Theta$ , gives precisely a (Connes-style [4]) noncommutative gauge theory. In this context, this theory can be rewritten in terms of the commutative one through a field redefinition known as the Seiberg-Witten map [3].

Under such a map gauge fields are written as an infinite series on the noncommutative parameter  $\Theta_{\mu\nu}$  ( $[\hat{x}_\mu, \hat{x}_\nu] = \Theta_{\mu\nu}$  where  $\hat{x}_\mu$  are the noncommutative coordinates of the spacetime). To every order (in  $\Theta$ ) the gauge field can be determined in terms of the usual (commutative) gauge field. The addition of higher-derivative terms do not spoil gauge invariance since the noncommutative gauge group action on the space of noncommutative connections, is such that the quotient is isomorphic to the corresponding quotient in the commutative case. This construction was realized explicitly for pure gauge fields. Later this construction was extended to the non-abelian case and coupled to matter fields [5, 6]. This proposal has been studied widely in the literature and used for a noncommutative gauge invariance extensions of the standard model and gravity (see for instance, [7] and [8]). In the case of the gravity such extension gave rise in a natural way to topological invariants such as the Euler characteristic and the signature with explicit computations in Ref. [9].

It is well known that the Wilson lines and loops are very useful in the description and computation of some non-perturbative aspects of gauge theory just as confinement [10]. In the context of string theory noncommutative Wilson loops have been studied within the correspondence gauge/gravity duality in Refs. [11, 12, 13, 14, 15]. In Ref. [16], in noncommutative gauge theories, Wilson lines were studied through the Schwinger-Dyson equations of correlation functions of Wilson lines. In field theory on a noncommutative two-dimensional torus the correlators of Wilson line operators were determined [17]. Also in the two-dimensional plane within a perturbative (in  $\Theta$  and in  $\frac{1}{\Theta}$ -expansion), the non-planar correlation functions of Wilson loops were obtained and the mixing UV/IR was consistently regularized [18, 19]. Wilson loops also were studied in the twisted covariant noncommutative field theory. In particular their correlation functions, Morita duality and the area preserving diffeomorphisms action, were examined in this context in Refs. [20, 21]. In Ref. [22], it was observed that in a gauge theory on the noncommutative plane, the area-preserving diffeomorphisms symmetry is non-perturbatively broken.

Noncommutative gauge theories have very striking topological and geometrical features. For instance, in string theory, in the absence of  $B$ -field, the instanton equation  $F^+ = 0$  is exact in  $\alpha'$  as the instanton shrinks, and this small instanton becomes a singularity of its moduli space. For non-vanishing  $B$ -field and non-small volume, the singularity is resolved and absent from the moduli space. This space defines precisely the moduli space of noncommutative instantons [23]. It is worth mentioning that recently in Ref. [24] there were discussed some geometrical implications of the Seiberg-Witten map in Chern-Simons and gravity. Some useful comments of noncommutative

Chern-Simons theory can be found in [25]. Some connections with the Seiberg-Witten equations are described in Refs. [26, 27, 28]. The consideration of some topological aspects of the noncommutative Wilson lines in the Seiberg-Witten limit is discussed in Ref. [29]. Another aspects of topological nature in noncommutative gauge theory were discussed in Ref. [30].

The Wilson loops possesses by themselves some interesting geometrical and topological properties e.g., in the abelian case, they can be regarded in terms of the linking number [31]. On the other hand Wilson loops are also useful for the description of knots and link invariants. In Ref. [32] the Jones polynomials were reproduced and generalized by computing correlation functions of products of Wilson loops using Chern-Simons action in the path integral formalism. Essentially the computations throw topological invariants since the action, observables and measure do not depends on the metric for Chern-Simons theories and their higher-dimensional generalization known as BF theories [33]. Linking numbers in BF theories were examined in Refs. [33, 34, 35].

In the present work we explore some geometrical and topological aspects based on the previous ideas but immersed in a noncommutative space using the Wilson lines constructed by means of the gauge field provided by the Seiberg-Witten map. As our main result, we will see that in this context is possible to establish a correspondence between the terms of a power series (in the non-commutative parameter  $\Theta$ ) series within the phase of a non-commutative Wilson loop. Each term of the series in  $\Theta$  has associated various linking numbers, at the  $n$ -th term of the expansion, there will arise  $6^n$  extra linking numbers. All these extra terms correspond to new homology cycles generated by the non-vanishing parameter  $\Theta$ .

The linking number is ordinarily a topological invariant, now the non-commutative linking numbers considered here, will represent a topological invariant of the corresponding more general non-commutative topology. Thus, the arising extra terms and their involved mathematical structures, deserve a detailed mathematical and physical interpretation and further analysis. In the present paper we will restrict ourselves to compute the first order non-commutative corrections to the linking numbers. This should be considered a first attempt of a description of the subject.

To explore how the modifications to topology of knots immersed on the non-commutative space we consider the abelian Jones-Witten polynomials in the path integral formalism which are given in terms of Wilson loops. We will show explicitly that the polynomials are changed due to noncommutativity, at least up to first order. However we should emphasize that, even at the first order, there will be a non-vanishing and non-trivial modification of the linking numbers due the noncommutative generalization of the notion of topology.

In order to explore our proposal in a more detailed way we consider the application of the non-commutative Wilson loops to the Aharonov-Bohm effect. Some literature on the non-commutative Aharonov-Bohm effect and its relation to the Landau levels can be found in Refs. [36, 37, 38, 39, 40, 41].

It is worth mentioning that the Wilson loops have been used in the description of some quantum theories of gravity. Some of these results can be found in Refs. [42, 43, 44, 45]. The results found in the present paper would be applied also for this kind of theories.

This paper is structured as follows: in section 2 we overview the Seiberg-Witten map and setting up the notation and conventions that we will follow along the paper. Section 3 is devoted to construct the noncommutative Wilson loops based on the gauge field of the Seiberg-Witten map. In sections 4 we introduce the linking numbers, first in order to explore the geometrical properties of noncommutative Wilson loops we use basic ideas of Poincaré duality and interpret higher order powers in the Seiberg-Witten expansion in terms of the arising new linking numbers. In section 5 we compute the abelian Jones-like polynomials using the path integral formalism through the Chern-Simons functional up to first order in the noncommutative parameter. As a physical application, in section 6 we explore the abelian noncommutative Aharonov-Bohm effect by means of the gauge field of the Seiberg-Witten map. The noncommutative Aharonov-Bohm effect is a very interesting physical example in which the noncommutativity could be relevant. It has known effects already described in the literature [36, 37, 38, 39, 40, 41]. Moreover it has a relation with the non-commutative Landau levels. We will find that our results are consistent with these mentioned results. Section 7 is devoted to final remarks.

## 2 The Seiberg-Witten map

Our aim is not to provide an extensive review on the Seiberg-Witten map [3], but only recall the relevant structure which will be needed in the following sections. Throughout this paper we will follow the notation and conventions introduced in reference [6].

We are interested in noncommutativity utilizing the Seiberg-Witten map [3]. This proposal was extended in [5, 6] for any gauge field coupled to matter. Below we present a brief description of this construction.

The central idea is to deform the algebraic structure of continuous spaces in particular the polynomials in  $N$  variables generated by powers of  $x^I$  where  $I = 0, \dots, N$ , which is a freely generated algebra  $\mathbb{C}[x^1, \dots, x^N]$ . Now consider the usual commutations relations between the coordinates

$$[x^\mu, x^\nu] = 0. \quad (1)$$

This algebraic structure will be deformed assuming

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \quad (2)$$

where  $\Theta^{\mu\nu} = -\Theta^{\nu\mu} \in \mathbb{R}$  i.e.  $\Theta^{\mu\nu}$  is the noncommutative parameter and  $\hat{x}$  are the noncommutative coordinates (as we can see this algebra is similar to the Heisenberg algebra in the phase space) for a formal description see [6].

Explicitly this modifies the way we multiply polynomials and in general functions over the noncommutative variables in terms of the commutative variables through the Moyal  $\star$ -product defined by:

$$f \star g(x) = \mu(e^{\frac{i}{2}\Theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g), \quad (3)$$

where  $\mu$  is the product map defined by  $\mu(f(x) \otimes g(x)) = f(x) \cdot g(x)$ .

In this context the gauge transformations of a matter field  $\Psi(x)$  is defined as

$$\delta_\Lambda \Psi(x) = i\Lambda \star \Psi(x), \quad (4)$$

$\Lambda(x)$  is the noncommutative gauge parameter which is Lie algebra-valued i.e.  $\Lambda(x) = \Lambda^a(x)T^a$ . Now let us compute explicitly the following variation of the field  $\Psi$

$$\begin{aligned} (\delta_{\Lambda_1} \delta_{\Lambda_2} - \delta_{\Lambda_2} \delta_{\Lambda_1}) \Psi &= (\Lambda_1 \star \Lambda_2 - \Lambda_2 \star \Lambda_1) \star \Psi \\ &= \frac{1}{2} ([\Lambda_1^{a\star}, \Lambda_2^b] \{T^a, T^b\} + \{\Lambda_1^{a\star}, \Lambda_2^b\} [T^a, T^b]) \star \Psi. \end{aligned} \quad (5)$$

It is worth mentioning that the fields are not Lie algebra-valued because not only we have commutators but also we have anticommutators. So the algebra that close both operations (commutators and anticommutators) is precisely the Universal Enveloping Algebra.

The covariant derivative is defined by

$$D_\mu^\star \Psi(x) = \partial_\mu \Psi(x) - i\hat{A}_\mu \star \Psi(x), \quad (6)$$

where  $\hat{A}_\mu$  is the noncommutative gauge field and transforms as

$$\delta_\Lambda \hat{A}_\mu = \partial_\mu \Lambda + i[\Lambda^\star, \hat{A}_\mu]. \quad (7)$$

We can see that these terms have infinite many degrees of freedom, but in Ref. [3] it was shown that all the higher-order terms depend only on the zeroth order terms (the commutative term) i.e. of the gauge parameter  $\Lambda^{(0)a}T^a$  and the gauge field  $A_\mu^{(0)a}T^a$ . Let us assume that the gauge parameter  $\Lambda_\alpha$  depends only on  $\alpha$  and  $A_\mu$  i.e. the gauge parameter and the gauge field respectively. With these assumptions bearing in mind, let us substitute it in the expression (5)

$$\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha + i(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) = i\Lambda_{-i[\alpha, \beta]}. \quad (8)$$

This expression could be solved perturbatively assuming an expansion in the parameter  $\Theta$  as  $\Lambda_\alpha = \Lambda_\alpha^{(0)} + \Lambda_\alpha^{(1)} + \dots$ , where  $\Lambda_\alpha^{(0)} = \alpha = \alpha^a T^a$ .

For example up to first order in  $\Theta$ , we find the following expression:

$$\delta_\alpha \Lambda_\beta^{(1)} - \delta_\beta \Lambda_\alpha^{(1)} - i[\alpha, \Lambda_\beta^{(1)}] - i[\Lambda_\alpha^{(1)}, \beta] - \Lambda_{-i[\alpha, \beta]}^{(1)} = -\frac{1}{2}\Theta^{\mu\nu}\{\partial_\mu \alpha, \partial_\nu \beta\}, \quad (9)$$

whose solution is  $\Lambda_\alpha^{(1)} = \alpha - \frac{1}{4}\Theta^{\mu\nu}\{A_\mu, \partial_\nu \alpha\}$ . With this expression we can compute in a similar way the expression for matter fields assuming the transformation at zero-th order  $\delta_\alpha \Psi^{(0)} = i\alpha \Psi^{(0)}$ .

For the gauge field we expand again in orders of  $\Theta$  as  $\hat{A} = A^{(0)} + A^{(1)} + A^{(2)} + \dots$  and substituting it into Eq. (7), up to first order in  $\Theta$ , we obtain

$$A_\mu^{(1)} = \frac{1}{4}\Theta^{\rho\sigma}(\{F_{\rho\mu}, A_\sigma\} - \{A_\rho, \partial_\sigma A_\mu\}). \quad (10)$$

Finally the field strength tensor is given by  $F_{\mu\nu}^* = i[D_\mu^*, d_\nu^*]$ , whose solution up to first order is

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\Theta^{\rho\sigma}(2\{F_{\rho\mu}, F_{\sigma\nu}\} + \{D_\rho F_{\mu\nu}, A_\sigma\} - \{A_\rho, \partial_\sigma F_{\mu\nu}\}). \quad (11)$$

Here we can identify  $F_{\mu\nu}^{(0)} = F_{\mu\nu}$  and  $F_{\mu\nu}^{(1)} = \frac{1}{4}\Theta^{\rho\sigma}(2\{F_{\rho\mu}, F_{\sigma\nu}\} + \{D_\rho F_{\mu\nu}, A_\sigma\} - \{A_\rho, \partial_\sigma F_{\mu\nu}\})$ .

### 3 Noncommutative Wilson Loops

The usual Wilson loop is given by the following expression,

$$W(C) = \text{Tr } P \exp \left( \frac{i}{\hbar} \int_C A \right), \quad (12)$$

where  $A = A_\mu^a t^a dx^\mu$ , with  $t^a$  being the Lie algebra generators, then the Wilson loop is an element of the Lie group and an element of the holonomy. An extension to the noncommutative case was proposed in a straightforward manner using the Moyal product in [26, 28]. In this work we make use of the gauge field given by the Seiberg-Witten map in order to construct the corresponding noncommutative Wilson loop, this amounts to change the connection  $A$  by  $\hat{A}$  which possesses an expansion in powers of  $\Theta$  given by  $\hat{A} = A^{(0)} + A^{(1)} + A^{(2)} + \dots$  and every  $A^{(i)}$  is expanded in terms of the usual 1-form basis i.e  $A^{(i)} = A_\mu^{(i)} dx^\mu$ , the Wilson loop corresponding to the Seiberg-Witten map is given by

$$\widehat{W}(C) = \text{Tr } P \exp_\star \left( \frac{i}{\hbar} \int_C \hat{A} \right), \quad (13)$$

For Chern-Simons (our case of interest) it was shown in Ref. [24] that the same formulae apply for the Seiberg-witten map. Further developments for higher-order computations of the Seiberg-Witten map can be found in [46].

The  $\star$ -exponential is defined as

$$\begin{aligned} \text{Tr } \exp_\star \left( \frac{i}{\hbar} \int_C \hat{A} \right) &= 1 + \frac{i}{\hbar} \int_C \text{Tr } \hat{A}(x) \\ &+ \frac{i^2}{\hbar^2 2!} \int_C \int_C \text{Tr } \left( \hat{A}(x) \star \hat{A}(y) \right) + \frac{i^3}{\hbar^3 3!} \int_C \int_C \int_C \text{Tr } \left( \hat{A}(x) \star \hat{A}(y) \star \hat{A}(z) \right) + \dots, \end{aligned} \quad (14)$$

where we are assuming the integration order  $x < y < z < \dots$ , with  $x, y, z, \dots \in \mathbb{C}$ .

#### 3.1 Abelian Case

For the abelian case the Wilson loop can be written as follows:

$$\widehat{W}(C) = \exp_{\star} \left( \frac{i}{\hbar} \int_C \widehat{A} \right) = \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(0)} \right) \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(1)} \right) \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(2)} \right) \dots \quad (15)$$

Then we can compute explicitly the Wilson loop up to second  $\Theta$  order. We can see that it is necessary to expand the first three exponentials

$$\widehat{W}(C) \approx \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(0)} \right) \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(1)} \right) \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(2)} \right). \quad (16)$$

It is easy to check the exponential  $\exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(j)} \right)$  up order  $2j + 1$  is given by

$$\begin{aligned} \exp_{\star} \left( \frac{i}{\hbar} \int_C A^{(j)} \right) &= \\ &1 + \frac{i}{\hbar} \int_C A^{(j)} + \frac{i^2}{2!\hbar^2} \int_{C \times C} A^{(j)} \star A^{(j)} + \frac{i^3}{3!\hbar^3} \int_{C \times C \times C} A^{(j)} \star A^{(j)} \star A^{(j)} + \dots \\ &\approx \exp \left( \frac{i}{\hbar} \int_C A^{(j)} \right) \left[ 1 + \frac{i^2}{2!\hbar^2} \frac{1}{2} \Theta^{\mu\nu} \partial_{\mu} \int_C A^{(j)} \cdot \partial_{\nu} \int_C A^{(j)} \right]. \end{aligned} \quad (17)$$

The second term vanishes in virtue of the abelianity and the skew symmetry of  $\Theta^{\mu\nu}$ , thus the second order Wilson loop is given by

$$\begin{aligned} \widehat{W}(C) &\approx \exp \left( \frac{i}{\hbar} \int_C A^{(0)} \right) \left[ 1 + \frac{i}{\hbar} \int_C A^{(1)} + \frac{i}{\hbar} \int_C A^{(2)} + \frac{1}{2!} \left( \frac{i}{\hbar} \right)^2 \int_C A^{(1)} \int_C A^{(1)} \right] \\ &= \exp \left( \frac{i}{\hbar} \alpha_0 \right) \left[ 1 + \frac{i}{\hbar} \Theta \alpha_1 + \frac{i}{\hbar} \Theta^2 \alpha_2 + \frac{1}{2!} \left( \frac{i}{\hbar} \right)^2 \Theta^2 \alpha_1 \alpha_1 \right]. \end{aligned} \quad (18)$$

As an example let us compute  $\widehat{W}(C)$  assuming a pure constant magnetic field along the  $z$ -axis  $\vec{B} = B_0 \hat{k}$  and  $C = S^1$  in the  $x - y$  plane, hence  $A_1^0 = -\frac{B_0}{2}y$  and  $A_2^0 = \frac{B_0}{2}x$ , thus

$$\alpha_0 = \int_C A_{\mu}^{(0)} dx^{\mu} = \frac{B_0}{2} (2\pi) = \phi, \quad (19)$$

$$\Theta \alpha_1 = \int_C A_{\mu}^{(1)} dx^{\mu} = \frac{3}{4} B_0^2 (2\pi) \Theta^{12} = \frac{3}{2\pi} \phi^2 \Theta^{12}, \quad (20)$$

$$\Theta^2 \alpha_2 = \int_C A_{\mu}^{(2)} dx^{\mu} = \frac{1}{2} B_0^3 (2\pi) (\Theta^{12})^2 = \frac{4}{(2\pi)^2} \phi^3 (\Theta^{12})^2, \quad (21)$$

where  $\phi$  is the flux due to the commutative field through the closed curve  $C$ . Now up to second order the noncommutative Wilson line is given by

$$\widehat{W}(C) \approx \exp \left( \frac{i}{\hbar} \alpha_0 \right) \left( 1 + \Theta \frac{i}{\hbar} \alpha_1 + \Theta^2 \left( \frac{i}{\hbar} \alpha_2 + \frac{1}{2} \left( \frac{i}{\hbar} \right)^2 \alpha_1^2 \right) \right)$$

$$= \exp \left( \frac{i}{\hbar} \frac{B_0}{2} (2\pi) \right) \left( 1 + \frac{i}{\hbar} \Theta^{12} \left[ \frac{3}{4} B_0 (2\pi) \right] + \frac{1}{2} \frac{i}{\hbar} (\Theta^{12})^2 \left[ (2\pi) B_0^3 + \frac{i}{\hbar} \left( \frac{3}{4} B_0^2 \right)^2 (2\pi)^2 \right] \right). \quad (22)$$

Therefore as we might expect, to observe noncommutativity effects assuming  $\Theta \ll 1$  we need the intensity of the magnetic field  $B_0$  to be large enough.

## 4 Linking Numbers and the Wilson Loops

It is well known that on a three manifold  $M = S^3$  or  $\mathbb{R}^3$  the magnetic field induced by a loop wire carrying a current, is proportional to the linking number between the loop and the magnetic lines, due the Biot-Savart law (which is essentially the Gauss linking number). This is written usually as

$$\Phi = \int_{\Sigma} F^{(0)} = \int_{\Sigma} dA^{(0)} = \int_C A^{(0)}, \quad (23)$$

where we assuming that  $C$  is a trivial homology 1-cycle (i.e. a trivial element in the first homology group). Thus there exists a 2-chain  $\Sigma$  (it could be regarded as a two-dimensional submanifold of  $\mathbb{R}^3$ ) such that  $C = \partial\Sigma$  is immersed on the three-dimensional Euclidean space. The connection  $A^{(0)} = A_{\mu}^{(0)} dx^{\mu}$  is the commutative (usual) magnetic potential 1-form. As we can see the last integral is the phase that appears on the commutative Wilson loop.

Since  $C$  is a trivial cycle by Poincaré duality there exists  $d\eta$  being  $\eta$  a 1-form. The mathematical interpretation assuming that  $C$  is a trivial 1-cycle and the flux is non-zero implies that  $A^{(0)}$  is not closed form, i.e.

$$\Phi = \int_M A^{(0)} \wedge d\eta = \int_M \eta \wedge dA^{(0)}. \quad (24)$$

In the previous integral we integrate out by parts in this way we can regard  $dA^{(0)}$  as the Poincaré dual of some trivial cycle  $C^{(0)}$  which is boundary of the 2-chain  $\Sigma^{(0)}$ . Hence this integral represents the linking number among the trivial cycles  $C$  and  $C^{(0)}$  being them the respective boundaries of the 2-cycles  $\Sigma$  and  $\Sigma^{(0)}$ .

Now we will consider the noncommutative case, then the phase in the noncommutative Wilson loop assuming the expansion on  $\Theta$  is given by

$$\int_C \hat{A} = \int_C A^{(0)} + \int_C A^{(1)} + \int_C A^{(2)} + \dots \quad (25)$$

Interpreting this in terms of linking numbers, the first term is the link number between  $C$  and  $C^{(0)}$ , the second term is interpreted in the following way. Let us consider from the Seiberg-Witten map the first order potential  $A_{\mu}^{(1)} = -\Theta^{\kappa\lambda} \frac{1}{2} A_{\kappa}^{(0)} (\partial_{\lambda} A_{\mu}^{(0)} + F_{\lambda\mu}^{(0)})$  that can be arranged as

$$A^{(1)} = \Theta^{\kappa\lambda} A_{\kappa\lambda}^{(1)} = \Theta^{\kappa\lambda} A_{\kappa\lambda\mu}^{(1)} dx^{\mu}. \quad (26)$$

In this way  $A^{(1)}$  can be regarded as the sum of six 1-forms  $A_{\kappa\lambda}^{(1)} = A_{\kappa\lambda\mu}^{(1)} dx^{\mu}$  (since we are working in  $\mathbb{R}^3$ )



$$\int_C A^{(1)} = \int_M A^{(1)} \wedge d\eta = \int_M \eta \wedge dA^{(1)} = \int_{C^{(1)}} \eta, \quad (27)$$

where  $C^{(1)}$  is a trivial 1-cycle. But explicitly the first integral  $\Theta^{\kappa\lambda} \int_C A_{\kappa\lambda}^{(1)}$  must be equal to  $\int_{C^{(1)}} \eta$  therefore it also will be expanded in  $\Theta$  then Eq. (27) reads

$$\Theta^{\kappa\lambda} \int_C A_{\kappa\lambda}^{(1)} = \Theta^{\kappa\lambda} \int_{C_{\kappa\lambda}^{(1)}} \eta. \quad (28)$$

This equation implies that  $C_{\kappa\lambda}^{(1)} = \partial\Sigma_{\kappa\lambda}^{(1)}$  is a trivial cycle, thus it can be regarded as the linking number between  $C$  and each  $C_{\kappa\lambda}^{(1)}$ , e.g. in the three dimensional Euclidean space

$$\Theta^{\kappa\lambda} \int_{C_{\kappa\lambda}^{(1)}} \eta = \Theta^{12} \left( \int_{C_{12}^{(1)}} \eta - \int_{C_{21}^{(1)}} \eta \right) + \Theta^{13} \left( \int_{C_{13}^{(1)}} \eta - \int_{C_{31}^{(1)}} \eta \right) + \Theta^{23} \left( \int_{C_{23}^{(1)}} \eta - \int_{C_{32}^{(1)}} \eta \right), \quad (29)$$

or

$$\Theta^{\kappa\lambda} \int_C A_{\kappa\lambda}^{(1)} = \Theta^{12} \int_C (A_{12}^{(1)} - A_{21}^{(1)}) + \Theta^{13} \int_C (A_{13}^{(1)} - A_{31}^{(1)}) + \Theta^{23} \int_C (A_{23}^{(1)} - A_{32}^{(1)}), \quad (30)$$

in general since  $A_{\kappa\lambda}^{(1)} \neq A_{\lambda\kappa}^{(1)}$ , then the linking number between  $C$  and  $C_{\kappa\lambda}^{(1)}$  is different from the one of  $C$  and  $C_{\lambda\kappa}^{(1)}$ . In figure (1) it is explicitly displayed the intersection between the  $C_{\lambda\kappa}^{(1)}$  1-cycles, represented in different colors by the simplest case when they are circles, and the original 1-cycle  $C$ .

Let us introduce the following notation:  $\text{Lk}(C_1, C_2)$  denotes the linking number between the homologically trivial 1-cycles  $C_1$  and  $C_2$ .

Now consider the second order term and reordering the expression  $A^{(2)}$  in the Seiberg-Witten map, it could be written in a similar way as in Eq. (26)

$$A^{(2)} = \Theta^{\kappa_1\lambda_1} \Theta^{\kappa_2\lambda_2} A_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)} dx^\mu. \quad (31)$$

Let  $A_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)} = A_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)} dx^\mu$ , in analogy to the first order term

$$\Theta^{\kappa_1\lambda_1} \Theta^{\kappa_2\lambda_2} \int_C A_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)} = \Theta^{\kappa_1\lambda_1} \Theta^{\kappa_2\lambda_2} \int_{C_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)}} \eta, \quad (32)$$

again here  $d\eta$  denotes is de Poincaré dual of  $\gamma$ . As we can see in general the total flux induced by the second order term is the sum of the 36 linking numbers  $\text{Lk}(C, C_{\kappa_1\lambda_1\kappa_2\lambda_2}^{(2)})$ .

Finally in general for the  $m$ -th order we rearrange the expression  $A^{(m)}$  from the Seiberg-Witten map we rewrite it as:

$$A^{(m)} = \Theta^{\kappa_1\lambda_1} \dots \Theta^{\kappa_m\lambda_m} A_{\kappa_1\lambda_1\dots\kappa_m\lambda_m}^{(m)} dx^\mu. \quad (33)$$

We define  $A_{\kappa_1\lambda_1\dots\kappa_m\lambda_m}^{(m)} = A_{\kappa_1\lambda_1\dots\kappa_m\lambda_m}^{(m)} dx^\mu$ . Thus we have

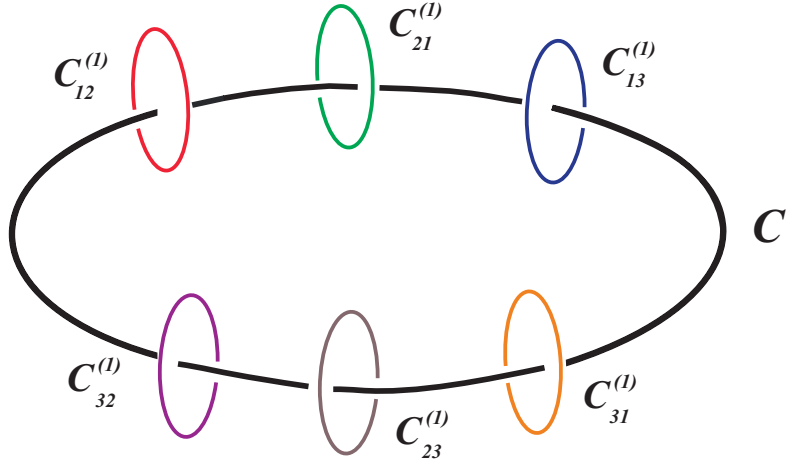


Figure 1: The figure accounts the computation at the first order in  $\Theta$  of the linking number. For the case of knots in  $\mathbb{R}^3$  it is observed at this order that the noncommutativity induces the existence of 6 different knots, represented in the simplest case by circles of different colors  $C_{\kappa\lambda}^{(1)}$ , that intersect the original trivial homology 1-cycle  $C$ . At the  $n$ -th order there will be  $6^n$  knots intersecting  $C$ . In the general case the knots  $C_{\kappa\lambda}^{(1)}$  would be truly knots, as for instance the trefoil or even more complicated knots.

$$\begin{aligned}\Theta^{\kappa_1 \lambda_1} \dots \Theta^{\kappa_m \lambda_m} \int_C A_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)} &= \Theta^{\kappa_1 \lambda_1} \dots \Theta^{\kappa_m \lambda_m} \int_{C_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}} \eta \\ &= \Theta^{\kappa_1 \lambda_1} \dots \Theta^{\kappa_m \lambda_m} \text{Lk}(C, C_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}),\end{aligned}\quad (34)$$

where  $C_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}$  are homologically trivial 1-cycles i.e. exist a 2-chain  $\Sigma_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}$ , such that  $C_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)} = \partial \Sigma_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}$ . Therefore in general the  $m$ -flux through  $C$  could be regarded as the sum of  $6^m$  linking numbers between  $C$  and  $C_{\kappa_1 \lambda_1 \dots \kappa_m \lambda_m}^{(m)}$  in  $\mathbb{R}^3$ .

## 5 The Jones-Witten-Like Invariants

The Jones polynomials in the Witten's path integral formulation are given by the correlation functions of Wilson loops

$$J_C = \langle W(C) \rangle = \frac{1}{N} \int \mathcal{D}A \exp \left[ \frac{i}{\hbar} k \int_M \text{Tr} \left( A \wedge dA + A \wedge A \wedge A \right) \right] \text{Tr}_R W(C), \quad (35)$$

where  $W(C) = P \exp[\frac{i}{\hbar} \int_C A]$  is the Wilson line,  $R$  is a representation of the gauge group,  $A$  is the gauge field,  $C$  is a knot (homologically trivial 1-cycle),  $M$  the Euclidean 3-dimensional space or the unitary 3-sphere and  $N$  is the normalization factor.

We will consider the simplest case when the gauge group is  $U(1)$  but consider the noncommutative Wilson loop (abelian case), and compute the first non-trivial correction which will arise at first order in  $\Theta$ .

First of all we substitute in the path integral  $A$  by  $\hat{A}$ , i.e the noncommutative Jones-Witten like invariants will depend explicitly of the noncommutative parameter (for the abelian case) read,

$$J_C(\Theta) = \langle \widehat{W}(C) \rangle = \frac{1}{N} \int \mathcal{D}\hat{A} \exp \left[ \frac{i}{\hbar} k \int_M \text{Tr} \left( \hat{A} \wedge d\hat{A} + \hat{A} \wedge \hat{A} \wedge \hat{A} \right) \right] \text{Tr}_R \widehat{W}(C). \quad (36)$$

Since we will consider the  $\Theta$ -expansion just up to first order, the term  $\hat{A} \wedge \hat{A} \wedge \hat{A}$  does not contribute (it contributes up to third order), then we get the following expression for the action

$$\begin{aligned}\exp \left[ \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + 2A^{(0)} \wedge dA^{(1)}) \right] \\ \approx \exp \left[ \frac{i}{\hbar} k \int_M A^{(0)} \wedge dA^{(0)} \right] \left[ 1 + \frac{i}{\hbar} 2k\Theta^{\kappa\lambda} \int_M A^{(0)} \wedge dA_{\kappa\lambda}^{(1)} \right],\end{aligned}\quad (37)$$

and the noncommutative Wilson line (13) up to first order in  $\Theta$  is

$$\widehat{W}(C) = \exp_\star \left( \frac{1}{\hbar} \int_C \hat{A} \right) \approx \exp \left( \frac{i}{\hbar} \int_C A^{(0)} \right) \left[ 1 + \frac{i}{\hbar} \Theta^{\kappa\lambda} \int_C A_{\kappa\lambda}^{(1)} \right]. \quad (38)$$

Then the functional integral up to first order (the first  $\Theta$  order Jones polynomials) is given by

$$\int \mathcal{D}A^{(0)} \exp \left( \frac{i}{\hbar} k \int_M A^{(0)} \wedge dA^{(0)} \right) \times \exp \left( \frac{i}{\hbar} \int_C A^{(0)} \right) \left[ 1 + \frac{i}{\hbar} \Theta^{\kappa\lambda} \int_C A_{\kappa\lambda}^{(1)} + \frac{i}{\hbar} 2k \Theta^{\kappa\lambda} \int_M A^{(0)} \wedge dA_{\kappa\lambda}^{(1)} \right]. \quad (39)$$

The integral is performed just on the commutative term  $A^{(0)}$  since every order can be written in term  $A^{(0)}$  in virtue of Seiberg-Witten map. Rewriting the previous expression we get

$$\int \mathcal{D}A^{(0)} \exp \left( \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + \frac{1}{k} A^{(0)} \wedge d\eta) \right) \times \left[ 1 + \frac{i}{\hbar} \Theta^{\kappa\lambda} \int_M A_{\kappa\lambda}^{(1)} \wedge d\eta + \frac{i}{\hbar} 2k \Theta^{\kappa\lambda} \int_M A^{(0)} \wedge dA_{\kappa\lambda}^{(1)} \right], \quad (40)$$

where  $d\eta$  is the Poincaré dual of the knot  $C$ . Next we need to compute these three integrals; the first one is the usual abelian Jones polynomial like for the knot  $C$ , whose evaluation is proportional to

$$\exp \left( - \frac{i}{4\hbar k} \int_M \eta \wedge d\eta \right), \quad (41)$$

where  $\int_M \eta \wedge d\eta$  is the Hopf invariant or self-linking number.

Now we consider the second integral which is rewritten as

$$\frac{i}{\hbar} \Theta^{\kappa\lambda} \int \mathcal{D}A^{(0)} \exp \left[ \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + \frac{1}{k} A^{(0)} \wedge d\eta) \right] \left[ \int_M A_{\kappa\lambda}^{(1)} \wedge d\eta \right]. \quad (42)$$

First of all, notice that any 1-form  $A^{(0)}$ , using the Hodge decomposition theorem, can be uniquely decomposed as  $A^{(0)} = A^{(0)_{\text{harm}}} + dB + \delta C$ , where  $A^{(0)_{\text{harm}}}$  is a harmonic 1-form (i.e. it is a solution to the laplacian  $\Delta_p = d\delta + \delta d$ ),  $B$  is a 0-form, and  $C$  is a 2-form. Here  $d$  the usual exterior derivative (which maps a  $p$ -form into a  $(p+1)$ -form) and  $\delta = (-1)^{n(p+1)+1} * d*$  their adjoint operator (which maps  $p$ -form into  $(p-1)$ -form). The  $*$  stands for the Hodge star operator which maps a  $p$ -form into an  $(n-p)$ -form. Also it is necessary to consider that  $*^2 = (-1)^{p(n-p)}$ .

In order to integrate out the path integral measure is decomposed into  $\mathcal{D}A^{(0)} = \mathcal{D}A^{(0)_{\text{harm}}} \mathcal{D}A^t \mathcal{D}A^l$  in virtue of the Hodge decomposition theorem, where  $A^t = dB$  and  $A^l = \delta C$  are the longitudinal and transversal parts of  $A^{(0)}$  respectively. It is well known that just the transversal parts contributes to the integration.

Moreover it is necessary to introduce some extra conventions. Let  $\Lambda_p$  the space of eigenforms of  $\Delta_p$  with non-vanishing eigenvalue  $\lambda^2$ . This space can be decomposed into  $\Lambda_p^l$  and  $\Lambda_p^t$  i.e. into the longitudinal and transversal parts. We also need to consider

the following maps:  $d : \Lambda_p^t \rightarrow \Lambda_{p+1}^t$ ,  $\delta : \Lambda_{p+1}^t \rightarrow \Lambda_p^t$  and an isomorphism given by  $* : \Lambda_p \rightarrow \Lambda_{n-p}$  among forms with the same eigenvalue  $\lambda^2$ . Given these maps finally we construct the following isomorphism:

$$\lambda^{-1} * d : \Lambda_p^t \rightarrow \Lambda_{n-p-1}^t. \quad (43)$$

Since we will consider  $M = S^3$  or  $\mathbb{R}^3$ , and 1-forms (this is  $n = 3$  and  $p = 1$ ), the last isomorphism is reduced to  $\lambda^{-1} * d : \Lambda_1^t \rightarrow \Lambda_1^t$ .

Now consider a basis of transverse 1-eigenforms  $\{A_i^t\}$  such that it satisfies the normalization condition  $\langle A_i^t | A_j^t \rangle = \int_M A_i^t \wedge * A_j^t = \delta_{ij}$  (inner product among 1-forms). Hence we expand the transversal part of  $A^{(0)}$  and  $\eta$  in term of this basis hence  $A^t = \sum_i a_i^{(0)} A_i^t$  and  $\eta^t = \sum_i \eta_i A_i^t$ . With the aid of the inner product,  $*^2 = +1$  for 2-forms in a Riemannian manifold and the isomorphism (43) the term in the exponential of the left-hand-side of (42) can be rewritten as follows

$$k \langle A^{(0)} | * dA^{(0)} \rangle + \langle A^{(0)} | * d\eta \rangle = k \sum_i \lambda_i \left( (a_i^{(0)})^2 + \frac{1}{k} a_i^{(0)} \eta_i \right), \quad (44)$$

where  $*dA^{(0)} = *d \sum_i a_i^{(0)} A_i^t = \sum_i a_i^{(0)} *dA_i^t = \sum_i a_i^{(0)} \lambda_i A_i^t$  and similarly for  $*d\eta$ .

In a similar spirit we compute the first order  $\Theta$  expansion of the Wilson line

$$\Theta^{\kappa\lambda} \int_M A_{\kappa\lambda}^{(1)} \wedge d\eta = \int_M A^{(1)} \wedge d\eta = \sum_i \lambda_i a_i^{(1)} \eta_i, \quad (45)$$

being  $A^{(1)} = \sum_\ell a_\ell^{(1)} A_\ell^t$ . Explicitly  $a_\ell^{(1)}$  can be written in terms of  $a^{(0)}$  since  $A_\mu^{(1)} = -\frac{1}{2} \Theta^{\kappa\lambda} A_\kappa^{(0)} (\partial_\lambda A_\mu^{(0)} + F_{\lambda\mu}^{(0)})$ . Using the eigenbasis expansion  $A^{(0)} = \sum_i a_i^{(0)} A_i^t$  and the following relation

$$dA_i^t = *^2 dA_i^t = * \lambda_i A_i^t = \frac{1}{2} \lambda_i A_{i\mu}^t \varepsilon_{\rho\sigma}^\mu dx^\rho \wedge dx^\sigma, \quad (46)$$

where we use the fact  $*^2 = 1$ . In virtue of the last expression we get the following useful expression for the partial derivatives of the gauge field components  $\partial_\rho A_{i\sigma} = \frac{1}{2} \lambda_i A_{i\mu}^t \varepsilon_{\rho\sigma}^\mu$ , hence we can rewrite  $A^{(1)}$  as:

$$A^{(1)} = \frac{3}{4} \Theta^{\kappa\lambda} \sum_{i,j} \lambda_j a_i^{(0)} a_j^{(0)} A_{i\kappa}^t A_{j\mu}^t \varepsilon_{\rho\lambda}^\mu dx^\rho. \quad (47)$$

Finally projecting out  $A^{(1)}$  into the transverse basis we get the following explicitly expression

$$a_\ell^{(1)} = \int_M A^{(1)} \wedge * A_\ell = \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j} \lambda_j a_i^{(0)} a_j^{(0)} [C_{ij\ell}]_{\kappa\lambda}, \quad (48)$$

where the  $C$ 's are defined by the following expression

$$[C_{ij\ell}]_{\kappa\lambda} = \int_M \varepsilon_{\gamma\lambda}^\mu \varepsilon_{\chi\rho}^\zeta A_{i\kappa}^t A_{j\mu}^t A_{\ell\zeta}^t \cdot dx^\gamma \wedge dx^\chi \wedge dx^\rho. \quad (49)$$

With these algebraic manipulations we can write Eq. (45) as

$$\int_M A^{(1)} \wedge d\eta = \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \eta_\ell \lambda_j \lambda_\ell a_i^{(0)} a_j^{(0)} [C_{ij\ell}]_{\kappa\lambda}. \quad (50)$$

Following a similar procedure we find

$$\int_M A^{(0)} \wedge dA^{(1)} = \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \lambda_j \lambda_\ell a_i^{(0)} a_j^{(0)} a_\ell^{(0)} [C_{ij\ell}]_{\kappa\lambda}. \quad (51)$$

Therefore Eq. (42) can be re-expressed in terms of the previous expansion then

$$\begin{aligned} \frac{1}{N} \int \mathcal{D}A^{(0)} \exp \left[ \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + \frac{1}{k} A^{(0)} \wedge d\eta) \right] &\cdot \left[ \frac{i}{\hbar} \int_M A^{(1)} \wedge d\eta \right] \\ &= \frac{1}{N} \int \prod_m da_m^{(0)} \exp \left[ \frac{i}{\hbar} k \left( \sum_m \lambda_m [(a_m^{(0)})^2 + \frac{1}{k} a_m^{(0)} \eta_m] \right) \right] \\ &\quad \times \left[ \frac{3}{8} \frac{i}{\hbar} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \eta_\ell \lambda_j \lambda_\ell a_i^{(0)} a_j^{(0)} [C_{ij\ell}]_{\kappa\lambda} \right], \quad (52) \end{aligned}$$

where  $N = \int \mathcal{D}A^{(0)} \exp \left[ ik \int_M A^{(0)} \wedge dA^{(0)} \right]$  is a normalization factor which is given by

$$N = \lim_{n \rightarrow \infty} \left( \frac{i\hbar\pi}{k} \right)^{\frac{n}{2}} \frac{1}{\sqrt{\det \Delta}}, \quad (53)$$

where  $\det \Delta = \prod_{m=1}^n \lambda_m$ .

In order to integrate out the expression (52), is convenient to rewrite it as follows

$$\frac{i}{\hbar} \Theta^{\kappa\lambda} \frac{3}{8} \sum_{\ell,m,i} \lambda_\ell \eta_i \lambda_i [C_{\ell mi}]_{\kappa\lambda} \int \prod_n da_n^{(0)} \exp \left( -\frac{1}{2} \sum_{i,j} a_i^{(0)} A_{ij} a_j^{(0)} + \sum_i a_i^{(0)} J_i \right) a_\ell^{(0)} a_m^{(0)}, \quad (54)$$

where  $A_{ij} = -\frac{2i}{\hbar} k \lambda_j \delta_{ij}$  and  $J_i = \frac{i}{\hbar} \lambda_i \eta_i$ , integrating out we obtain

$$\begin{aligned} \frac{1}{N} \int \mathcal{D}A^{(0)} \exp \left[ \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + \frac{1}{k} A^{(0)} \wedge d\eta) \right] &\left[ i \int_M A^{(1)} \wedge d\eta \right] \\ &= \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \lambda_j \lambda_\ell [C_{ij\ell}]_{\kappa\lambda} \exp \left( -\frac{i}{4k\hbar} \int_M \eta \wedge d\eta \right) \left( \frac{i}{4\hbar k^2} \eta_i \eta_j \eta_\ell - \frac{1}{2k} \frac{\delta_{ij} \eta_\ell}{\lambda_j} \right). \quad (55) \end{aligned}$$

Finally the third contribution in (40) associated to the factor  $\int_M A^{(0)} \wedge dA^{(1)}$ , then the term to compute is the following

$$\begin{aligned}
& \frac{2ik}{\hbar N} \int \mathcal{D}A^{(0)} \exp \left[ \frac{i}{\hbar} k \int_M (A^{(0)} \wedge dA^{(0)} + \frac{1}{k} A^{(0)} \wedge d\eta) \right] \left[ \int_M A^{(0)} \wedge dA^{(1)} \right] \\
&= \frac{2k}{N} \int \prod_m da_m^{(0)} \exp \left[ \frac{i}{\hbar} k \left( \sum_m \lambda_m [(a_m^{(0)})^2 + \frac{1}{k} a_m^{(0)} \eta_m] \right) \right] \\
&\quad \times \left[ \frac{3}{8} \frac{i}{\hbar} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \lambda_j \lambda_\ell a_i^0 a_j^0 a_\ell^{(0)} [C_{ij\ell}]_{\kappa\lambda} \right]. \quad (56)
\end{aligned}$$

The integral has the form in terms of  $A_{ij}$  and  $J_i$ , then we get the expression

$$\begin{aligned}
& \frac{2ik}{\hbar} \int \prod_m da_m^{(0)} \exp \left[ \frac{i}{\hbar} k \left( \sum_m \lambda_m \left[ (a_m^{(0)})^2 + \frac{1}{k} a_m^{(0)} \eta_m \right] \right) \right] \cdot \left[ \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \lambda_j \lambda_\ell a_i^0 a_j^0 a_\ell^{(0)} [C_{ij\ell}]_{\kappa\lambda} \right] \\
&= \frac{3}{8} \Theta^{\kappa\lambda} \sum_{i,j,\ell} \lambda_j \lambda_\ell [C_{ij\ell}]_{\kappa\lambda} \exp \left( -\frac{i}{4\hbar k} \int_M \eta \wedge d\eta \right) \\
&\quad \times \left[ -\frac{i}{4\hbar k^2} \eta_i \eta_j \eta_\ell + \frac{1}{2k} \left( \frac{\delta_{i\ell} \eta_j}{\lambda_i} + \frac{\delta_{j\ell} \eta_i}{\lambda_j} + \frac{\delta_{ij} \eta_\ell}{\lambda_j} \right) \right]. \quad (57)
\end{aligned}$$

Thus, the total contribution to the Jones-like polynomial up to first order in  $\Theta$  defined in (40) is obtained from the superposition of equations (41), (56) and (57), yielding to the expressions

$$J_C(\Theta) = \exp \left( -\frac{i}{4\hbar k} \int_M \eta \wedge d\eta \right) \left[ 1 + \frac{3}{16k} \Theta^{\kappa\lambda} \sum_{i,j} ([C_{iji}]_{\kappa\lambda} \lambda_j \eta_j + [C_{ijj}]_{\kappa\lambda} \lambda_j \eta_i) \right]. \quad (58)$$

As we can see the zero-th order is the usual  $U(1)$  "Jones" polynomial, where  $\eta$  is the Poincaré dual of  $C$  and the following term is a polynomial over  $\Theta$  related to the noncommutativity up to first order.

## 6 Noncommutative Aharonov-Bohm Effect

This section is devoted to explore some physical applications of the noncommutative Wilson loops and linking numbers, in particular we consider the Aharonov-Bohm effect which is a very good arena to test the physical ideas and extract visible effects. We are aware that this subject is present in the literature, see for instance [36, 37, 38, 39, 40, 41]. We shall see that our results will be agree with their results. Aharonov-Bohm effect consists of an electron beam through a double slit in presence of a small impenetrable solenoid which has a non-vanishing constant magnetic field inside (and therefore a non-vanishing vector potential  $A_\mu^{(0)}$ ). Outside the solenoid the magnetic field is zero, but not the potential, thus an interference pattern is observed due to the fact that the

vector potential is non-vanishing. The effect is measured as a phase factor in the wave function.

In the usual Aharonov-Bohm effect it is assumed that the wave function is of the form  $\Phi = \phi \exp(F)$ . Under this ansatz one finds the value of  $F$  by means of the covariant derivative  $D_j \exp(F) = k_j \exp(F)$ . In our case, we will assume that the corresponding noncommutative function  $\widehat{F}$  can be expanded in terms a noncommutative expansion in terms of the noncommutative parameter  $\Theta$ ; given by the ansatz [39]  $\widehat{F} = F^{(0)} + F^{(1)} + F^{(2)} + \dots$ . We will also determine  $\widehat{F}$  using the covariant derivative defined in the first section. Thus we have

$$D_j \star \exp_\star(\widehat{F}) = k_j \exp_\star(\widehat{F}), \quad (59)$$

considering just the expansion up to second order, it can be written up to second order as

$$\begin{aligned} & \partial_j \left[ \exp(F^{(0)}) \left( 1 + F^{(1)} + F^{(2)} + \frac{1}{2}(F^{(1)})^2 \right) \right] - i(A_j^{(0)} + A_j^{(1)} + A_j^{(2)}) \exp(F^{(0)}) \\ & \times \left( 1 + F^{(1)} + F^{(2)} + \frac{1}{2}(F^{(1)})^2 \right) + \frac{1}{2} \Theta^{\kappa\lambda} [\partial_\kappa (A_j^{(0)} + A_j^{(1)})] [\partial_\lambda (F^{(0)} + F^{(1)})] = k_j. \end{aligned} \quad (60)$$

Thus we obtain the following equations at each order

$$\partial_j F^{(0)} - i A_j^{(0)} = k_j, \quad (61)$$

$$\partial_j F^{(1)} - i A_j^{(1)} + \frac{1}{2} \Theta^{\kappa\lambda} (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(0)}) = 0, \quad (62)$$

$$\partial_j F^{(2)} - i A_j^{(2)} + \frac{1}{2} \Theta^{\kappa\lambda} (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(1)}) + \frac{1}{2} \Theta^{\kappa\lambda} (\partial_\kappa A_j^{(1)}) (\partial_\lambda F^{(0)}) = 0. \quad (63)$$

We can solve  $F^{(0)}$  in terms of  $A_j^{(0)}$ , then we solve for  $F^{(1)}$  in terms of  $A^{(1)}$  and  $F^{(0)}$ , and son on. In this way we find at each order the  $F^{(i)}$ 's, explicitly

$$\begin{aligned} F^{(0)} &= k_j x^j + i \int_C A_j^{(0)} dx^j, \\ F^{(1)} &= i \int_C A_j^{(1)} dx^j - \frac{1}{2} \Theta^{\kappa\lambda} \int_C (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(0)}) dx^j, \\ F^{(2)} &= i \int_C A_j^{(2)} dx^j - \frac{1}{2} \Theta^{\kappa\lambda} \int_C (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(1)}) dx^j - \frac{1}{2} \Theta^{\kappa\lambda} \int_C (\partial_\kappa A_j^{(1)}) (\partial_\lambda F^{(0)}). \end{aligned} \quad (64)$$

Finally the expression for  $\widehat{F}$  up to second order is given by

$$\begin{aligned} \widehat{F} &= k_j x^j + i \int_C (A_j^{(0)} + A_j^{(1)} + A_j^{(2)}) dx^j - \frac{1}{2} \Theta^{\kappa\lambda} \int_C \left[ (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(0)}) \right. \\ & \quad \left. + (\partial_\kappa A_j^{(0)}) (\partial_\lambda F^{(1)}) + (\partial_\kappa A_j^{(1)}) (\partial_\lambda F^{(0)}) \right] dx^j. \end{aligned} \quad (65)$$



From this equation we easily recognize the second term which correspond to the second order expansion of the noncommutative Wilson loop, and the following terms are the noncommutative corrections to the holonomy.

In the usual Aharonov-Bohm effect the potential outside the solenoid is given by

$$A_1^{(0)} = -\frac{x_2}{x_1^2 + x_2^2}, \quad A_2^{(0)} = \frac{x_1}{x_1^2 + x_2^2}, \quad (66)$$

for reference the expansion up to first and second order of this potential is

$$A_1^{(1)} = \frac{1}{2}\Theta^{12}\frac{x_2}{(x_1^2 + x_2^2)^2}, \quad A_2^{(1)} = -\frac{1}{2}\Theta^{12}\frac{x_1}{(x_1^2 + x_2^2)^2}, \quad (67)$$

$$A_1^{(2)} = -\frac{1}{4}(\Theta^{12})^2\frac{2x_2^3 + x_1^2x_2}{(x_1^2 + x_2^2)^4}, \quad A_2^{(2)} = \frac{1}{2}(\Theta^{12})^2\frac{2x_1^3 + x_1x_2^2}{(x_1^2 + x_2^2)^4}. \quad (68)$$

Meanwhile inside the components of the solenoid are

$$A_1^{(0)} = -\frac{B}{2}x_2, \quad A_2^{(0)} = \frac{B}{2}x_1, \quad (69)$$

where  $B$  is the magnitude of a constant magnetic field. At first and second order the potential is given by:

$$A_1^{(1)} = -\frac{3}{8}B^2\Theta^{12}x_2, \quad A_2^{(1)} = \frac{3}{8}B^2\Theta^{12}x_1, \quad (70)$$

$$A_1^{(2)} = -\frac{5}{16}B^3(\Theta^{12})^2x_2, \quad A_2^{(2)} = \frac{5}{16}B^3(\Theta^{12})^2x_1. \quad (71)$$

In analogy to the usual Aharonov-Bohm effect the phase difference is modified proportional to the flux through the solenoid. Then substituting equations (69, 70, 71) in (65) we obtain the wave function's phase of the non-commutative Aharonov-Bohm effect,

$$\hat{F} = k_j x^j + i\pi B r^2 + i\frac{1}{2}\Theta^{12}\pi B r^2 + \frac{5}{8}i(\Theta^{12})^2\pi B r^2, \quad (72)$$

where  $\Phi = \pi r^2 B$  is the flux of the magnetic field trough the solenoid (whose radius is  $r$ ). Finally the correction to the phase due the noncommutativity is

$$\exp(\hat{F}) \approx \exp\left(\frac{ie}{\hbar}k_j x^j\right) \exp\left[\frac{ie}{\hbar}\Phi + \frac{3ie}{4\hbar}\Phi\Theta^{12} + \frac{5ie}{8\hbar}\Phi(\Theta^{12})^2\right]. \quad (73)$$

As we can see the first term in the imaginary exponential is the standard holonomy (commutative), the second and third terms are correction to the holonomy due the noncommutativity up to second order given in terms of the usual flux and the noncommutative parameter.

Now we proceed to show that the phase of the equation (73) is related to the non-commutative Landau levels showed in Ref. [47]. To check closely this affirmation let us

make some explicit computations. First of all we define the non-commutative canonical momentum  $\widehat{\Pi}_\mu$  as usual but changing the usual gauge field  $A_\mu$  by its noncommutative one  $\widehat{A}_\mu$ , i.e.

$$\widehat{\Pi}_\mu = p_\mu + e\widehat{A}_\mu = p_\mu + e(A_\mu^{(0)} + A_\mu^{(1)} + A_\mu^{(2)} + \dots). \quad (74)$$

In addition, the consideration of quantum mechanical systems in a non-commutative space leads to the following commutation relations

$$[p_\mu, p_\nu] = 0, \quad [x_\mu, x_\nu] = 0, \quad [p_\mu, x_\nu] = -i\hbar\delta_{\mu\nu}. \quad (75)$$

With the aid of previous relations and baring in mind the expressions (69 – 71) let us compute the following commutator up to first order in  $\Theta$ , this is given by

$$[\widehat{\Pi}_1, \widehat{\Pi}_2] = -ie\hbar B(1 + \frac{3}{4}\Theta^{12}). \quad (76)$$

Let us construct the creation-annihilation operators as

$$a = \frac{\widehat{\Pi}_1 - i\widehat{\Pi}_2}{\sqrt{(2e\hbar B)(1 + \frac{3}{4}\Theta^{12})}}, \quad a^\dagger = \frac{\widehat{\Pi}_1 + i\widehat{\Pi}_2}{\sqrt{(2e\hbar B)(1 + \frac{3}{4}\Theta^{12})}}, \quad (77)$$

which satisfy the usual relation  $[a, a^\dagger] = 1$ . Now we will consider the hamiltonian  $H = \frac{1}{2m}(\Pi_1^2 + \Pi_2^2)$  and rewrite it in terms of  $a$  and  $a^\dagger$

$$H = \frac{eB\hbar}{2m}(1 + \frac{3}{4}\Theta^{12})(aa^\dagger + a^\dagger a), \quad (78)$$

where the factor  $\omega = \frac{eB}{2m}(1 + \frac{3}{4}\Theta^{12})$  is the angular frequency. Using the normal ordering we finally get

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}). \quad (79)$$

The examination of the phase in (73) up to first order, it clearly contains the frequency of the non-commutative Landau levels.

To estimate the order of  $\Theta$  we can compare with the results in reference [39], where they don't use the Seiberg-Witten map. Similarly as Ref. [39], it is possible to formulate the problem of scattering charged particles in an effective radial potential. The computation is exactly the same. Thus, at the first order we obtain the same result than in [39] if we make the following substitution:  $\Theta$  by  $-3\Theta^{12}$ . Then we conclude that the non-commutative parameter is precisely of the same order of magnitude as they estimated  $\Theta^{12} \approx [10 \text{ Tev}]^{-2}$ .

## 7 Final Remarks

In this paper we proposed to use the gauge field provided by the Seiberg-Witten map to construct study noncommutative Wilson loops. After a brief account on noncommutative Wilson loops, we study abelian Chern-Simons theory on a three dimensional manifold. It was shown that the effect of noncommutativity is the appearance of  $6^n$

new knots at the  $n$ -th order of the Seiberg-Witten expansion. These knots constitute trivial homology cycles which are Poincaré dual to the high-order Seiberg-Witten potentials of the expansion. Moreover the linking number at  $n$ -th order of a standard 1-cycle with the multiple Poincaré dual of the gauge fields is shown to be written as the sum of the linking number of this 1-cycle with the multiple Poincaré dual of the Seiberg-Witten gauge fields at this order (34). The generalization to higher dimensions can be done straightforwardly.

Furthermore as a topological application of the noncommutative gauge theories and Wilson loops in the abelian case; by using the path integral formalism and the Chern-Simons theory we computed the first order and non-vanishing correction due to the noncommutativity of the abelian Jones-like polynomials (58). This term is also of a topological nature and it represents a new noncommutative topological effect of link invariants of three-manifolds.

Furthermore as a physical application we compute explicitly the abelian Aharonov-Bohm effect in  $\mathbb{R}^3$  and calculate the wave function up to second order in the non-commutativity parameter (73). It results in the usual wave function in terms of the eigenvalue  $k_j$  in the real exponential and in the imaginary part appears the contribution of the usual flux (commutative) and a second order contribution which is proportional to the square of the flux. We discuss the relation of the Aharonov-Bohm effect and the Landau levels in the non-commutative context. These results are found to agree with those found in Refs. [36, 37, 38, 39, 40, 41]. In particular, the parameter  $\Theta$  is constrained and it basically coincides with that of Ref. [39].

It could be interesting to explore some geometrical aspects we might extend the linking number between knots in the three dimensional Euclidean space in a noncommutative sense and explore the different orders of the gauge potential and how they could give new information about linking numbers.

Moreover we would like to extend the present noncommutative ideas to higher dimensional theories, through a  $BF$  theory since it is a higher-dimensional generalization of Chern-Simons theory and the Wilson line will be interpreted in terms of linking numbers between higher-dimensional objects [33, 34].

Wilson loops for the spin connection are very important in some theories of quantum gravity [42, 43, 44, 45]. It is worth to study a noncommutative version of these models by using the noncommutative Wilson loops described here. Some of this work is left for future calculations.

As a further step we are interested in the natural extension to the non-abelian case, where we will deal with two expansions: the first one focusing in the noncommutative parameter and the second one due the non-abelianity of the Chern-Simons theory. Also we will study the physical implications using non-abelian Aharonov-Bohm effect [48]. For future work we leave the problem of studying the generalization of noncommutative Aharonov-Bohm effect and its associated Landau levels for this non-abelian case.

## Acknowledgments

The work of H. G-C. was partially supported by the CONACyT research grant: 128761. The work of O.O. is supported by a PROMEP, CONACyT and UG grants. In addition the work of R. S-S. was partially supported by a PROMEP and CONACyT postdoctoral fellowship.

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